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Moving contact lines and rivulet instabilities. Part 2. Long waves on flat rivulets

By RALPH H. WEILAND
Department of Chemical Engineering, Clarkson College of Technology, Potsdam, New York 13676

AND STEPHEN H. DAVIS
Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois 60201

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A rivulet is a narrow stream of liquid located on a solid surface and sharing a curved interface with the surrounding gas. Long-wave instabilities are investigated for flat rivulets on a vertical wall. Three cases of contact-line conditions are investigated: (i) fixed contact lines, (ii) moving contact lines having fixed contact angles and (iii) moving contact lines whose contact angles vary smoothly with contact-line speeds. For case (i), the straight, unidirectional rivulet is stable below a critical Reynolds number. For case (ii), the rivulet is unconditionally unstable. For case (iii), the variation, $G_1 \equiv [d\alpha/dU_{CL}]_{U_{CL}=0}$, in contact angle α with contact-line speed U_{CL} stabilizes the rivulet and if G_1 is large enough can completely stabilize the flow. The analysis lends support to the idea that contact-angle steepening with contact-line motion is a purely dissipative process.

1. Introduction

A rivulet is a narrow stream of liquid flowing along a solid surface and sharing an interface with the surrounding fluid. The flow within the rivulet is driven by the component of gravity along the solid surface as shown in figure 1. Rivulets are often seen on automobile windshields and on the walls of showers. They are frequently formed when uniform films break up and during condensation processes.

Rivulets display a large variety of intriguing instability phenomena. Kern (1969, 1971) sees the break-up of straight rivulets into droplets, rivulet meandering and the transition of rivulet flow from laminar to turbulent regimes. Large-amplitude surface waves are apparent in many situations. Our own preliminary experiments (Culkin 1979) show these together with more intricate phenomena. Clearly, the state of the fluid motion is important in assessing the rate of heat and mass transport in these systems and thus precise stability criteria are required.

The feature of rivulets that makes them most interesting and also so difficult to analyse is the existence of their contact lines. A contact line is the geometric curve formed by the intersection with the solid of an interface between two immiscible fluids. The rivulet sketched in figure 1 shows two such lines. When a contact line moves and the no-slip condition is enforced on all fluid–solid boundaries, an infinite-force singularity is present at the contact line (Dussan V. & Davis 1974). This is due to the

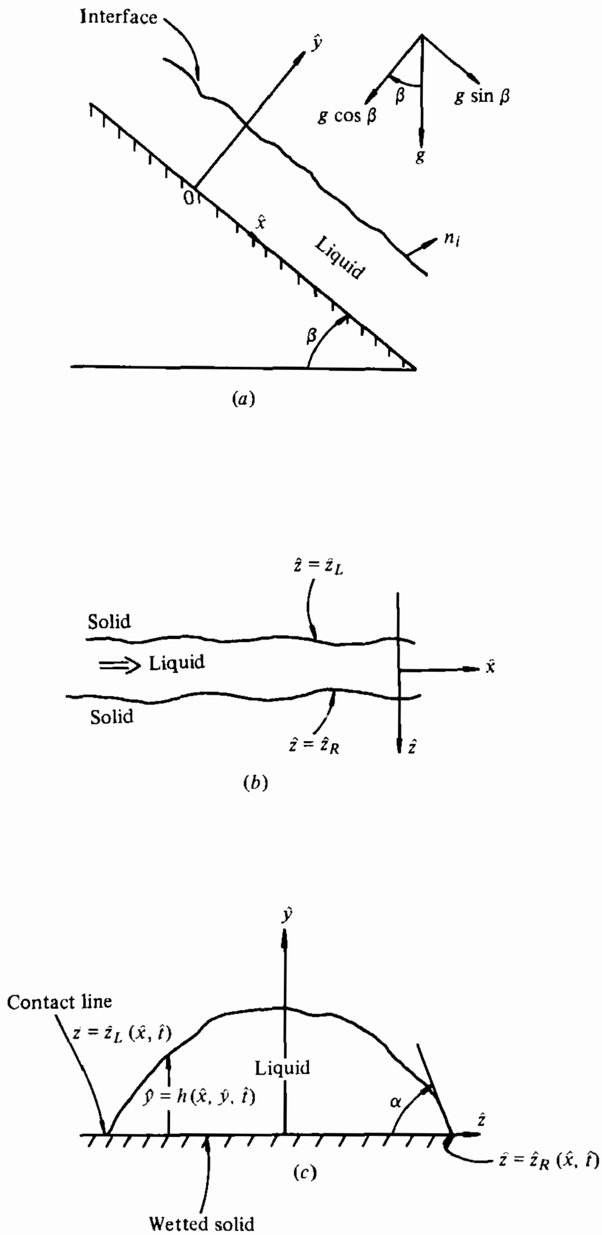

FIGURE 1. Sketch of a rivulet: (*a*) side view, (*b*) plan view, (*c*) front view.

kinematics of mutual displacement. If the local details of the flow are of interest, as they are in rivulet instabilities, the singularity must be eliminated. This can be done by allowing *effective slip* near the contact line. This route has been used with success in several analyses involving mutual displacement of one viscous fluid by another (Dussan V. 1976; Hocking 1977; Huh & Mason 1977). These areas are ably reviewed by Dussan V. (1979).

The rivulet can thus be seen to have interent fluid-dynamical interest. It possesses

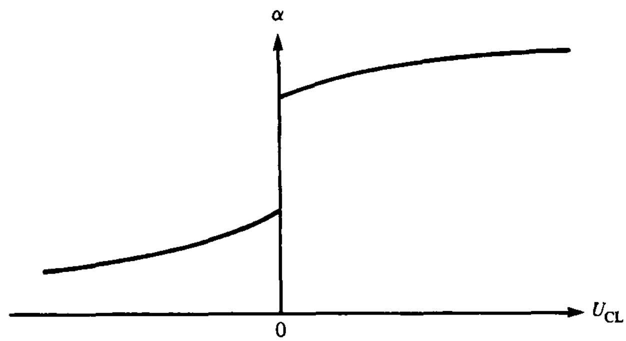

FIGURE 2. Sketch of experimental results of contact angle α *vs.* contact-line speed U_{CL}. $U_{CL} > 0$ denotes liquid displacing gas; $U_{CL} < 0$ denotes gas displacing liquid.

free boundaries and moving contact lines, and it displays a large variety of instability phenomena. Furthermore, it can play the role of a vehicle for the study of moving-contact-line boundary conditions by assessment of their effects on the gross instability characteristics of the system.

The only previous instability analysis on rivulet instabilities is given by Davis (1980). He considers a small, *static* rivulet on a horizontal plate. Owing to the curvature of the free surface, the fluid configuration is susceptible to capillary instabilities of Rayleigh (1879)-type. However, in contrast to the Rayleigh jet, the rivulet has solid–liquid contact and hence contact lines. Davis (1980) considers three types of contact-line conditions, (i) fixed contact lines, (ii) fixed contact angles and (iii) contact angles that vary smoothly with contact-line speeds. Davis then manipulates the linearized stability equations into a balance equation for kinetic energy, which has the form of a damped linear-harmonic-oscillator equation of the form

$$E\sigma^2 + \Phi\sigma + I = 0, \tag{1.1}$$

where σ is the linear-stability-theory growth rate, E is the kinetic energy of a small disturbance, Φ is the viscous dissipation and I is the interfacial energy due to surface tension. The above is the situation for case (i) of fixed contact lines. Stability follows for $I > 0$ and this is interpreted in terms of contact angle and disturbance wavenumber to give explicit stability predictions. For case (ii) where the contact angle is fixed upon disturbance but where the contact line moves, the effective slippage allowed at the liquid–solid surface becomes included in the effective dissipation Φ. Again, $I > 0$ implies stability and explicit results are obtained. Of most interest is case (iii), where there is a dynamic contact-line condition; it is found that the steepening of the contact angle with contact-line speed is a purely dissipative effect in that it contributes only to Φ and not to the functional I. Hence, the stability conditions for cases (ii) and (iii) are identical. Davis (1980) does discuss aspects of contact-angle hysteresis, the common situation shown in figure 2.

In the present work we pose the linearized hydrodynamic stability theory for a *dynamic* rivulet flow down a vertical planar wall. We consider the same three cases (i), (ii), and (iii) of contact-line conditions as posed by Davis (1980). Explicit stability results are obtained for travelling-wave disturbances whose wavelength λ is much longer than the maximum depth h_0 of the undisturbed rivulet, $k \equiv 2\pi h_0/\lambda \ll 1$, and where the rivulet is very flat, $\delta \equiv h_0/L \ll 1$, where L is the half-width of the undisturbed rivulet. We presume that $k \ll \delta$, though other cases are easily handled. This theory is analogous to that of Yih (1963) for film flow if $\delta \equiv 0$ and no contact lines are present. It is somewhat similar to the long-wave analysis of Rothrock (1968) who looked at pendant rivulets having no contact lines. However, our procedure is different in detail.

The results of our analysis are as follows: (i) When the contact lines are fixed, there is a critical Reynolds number R_C below which the rivulet is stable. R_C is much larger than that of film flow even if $\delta \to 0$. (ii) When the contact lines are free to move but subject to fixed contact angle, the rivulet is unconditionally unstable. (iii) When the contact lines are free to move, subject to their contact angles being smooth functions of contact-line speed, then the rivulet is always *more stable* than that of case (ii). In fact the degree of stabilization is proportional to $G_1 \equiv [d\alpha/dU_{CL}]_{U_{CL}=0}$, the slope of the curve of contact angle α *versus* contact-line speed U_{CL}. In fact when $G_1 \to \infty$ one approaches the case of contact-angle hysteresis, which then seems to give a rivulet in a very stable configuration. These results will be discussed later. Finally, the conclusion of Davis (1980), that the increase in contact angle with contact-line speed is a pure dissipative process, is given further support.

2. Formulation

A long, smooth flat plate is inclined at an angle β to the horizontal. A narrow stream of liquid, a rivulet, flows down the plane as shown in figure 1. This Newtonian liquid has constant density ρ and constant viscosity μ. The flow is driven by the component $g \sin\beta$ of gravity along the plate and the system is isothermal. The surrounding fluid is a passive gas that applies a constant atmospheric pressure on the liquid–gas interface.

The governing equations for this system are the Navier–Stokes equations and the equation of continuity:

$$\rho(\hat{u}_{i\hat{t}} + \hat{u}_j \hat{u}_{i,j}) = \hat{\sigma}_{ij,j} + \rho \hat{F}_i \tag{2.1a}$$

and

$$\hat{u}_{i,i} = 0, \tag{2.1b}$$

where $\hat{u}_i$ is the velocity vector, $(\hat{u}_i) = (\hat{u}, \hat{v}, \hat{w})$, $\hat{\sigma}_{ij}$ is the stress tensor,

$$\hat{\sigma}_{ij} = -\hat{p}\delta_{ij} + \mu(\hat{u}_{i,j} + \hat{u}_{j,i}), \tag{2.1c}$$

and $\hat{F}_i$ is the body force per unit mass due to gravity,

$$(\hat{F}_i) = g(\sin\beta, \ -\cos\beta, 0). \tag{2.1d}$$

The summation convention is assumed.

Equations (2.1) are referred to a right-handed Cartesian co-ordinate system, shown in figure 1, whose origin is on the plate, whose $\hat{x}$ axis points down the plane and whose $\hat{y}$ axis points normal to the plate into the liquid.

The boundary conditions appropriate to the liquid–gas interface, at $\hat{y} = \hat{\eta}(\hat{x}, \hat{z}, \hat{t})$, are the kinematic condition,

$$\hat{v} = \hat{\eta}_{\hat{t}} + \hat{u}\hat{\eta}_{\hat{x}} + \hat{w}\hat{\eta}_{\hat{z}} \tag{2.2a}$$

and the stress jump appropriate to an uncontaminated interface having constant surface tension T,

$$[\![\hat{\sigma}_{ij}]\!]\hat{n}_j = 2\hat{h}T\hat{n}_i. \tag{2.2b}$$

Here $\hat{\mathbf{n}}$ is the unit outward normal vector to the interface,

$$\hat{\mathbf{n}} = (-\hat{\eta}_{\hat{x}}, 1, -\hat{\eta}_{\hat{z}})(1+\hat{\eta}_{\hat{x}}^2+\hat{\eta}_{\hat{z}}^2)^{-\frac{1}{2}} \tag{2.2c}$$

and $\hat{h}$ is the mean curvature of the surface,

$$2\hat{h} = [\hat{\eta}_{\hat{x}\hat{x}}(1+\hat{\eta}_{\hat{z}}^2) - 2\hat{\eta}_{\hat{x}}\hat{\eta}_{\hat{z}}\hat{\eta}_{\hat{x}\hat{z}} + \hat{\eta}_{\hat{z}\hat{z}}(1+\hat{\eta}_{\hat{x}}^2)](1+\hat{\eta}_{\hat{x}}^2+\hat{\eta}_{\hat{z}}^2)^{-\frac{3}{2}}. \tag{2.2d}$$

It will be useful in what follows to define two orthogonal unit tangent vectors $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ as follows:

$$\mathbf{t}^{(1)} = (0, \hat{\eta}_{\hat{z}}, 1)(1+\hat{\eta}_{\hat{z}}^2)^{-\frac{1}{2}} \tag{2.2e}$$

and

$$\mathbf{t}^{(2)} = (1+\hat{\eta}_{\hat{z}}^2, \hat{\eta}_{\hat{x}}, -\hat{\eta}_{\hat{x}}\hat{\eta}_{\hat{z}})(1+\hat{\eta}_{\hat{z}}^2)^{-\frac{1}{2}}(1+\hat{\eta}_{\hat{x}}^2+\hat{\eta}_{\hat{z}}^2)^{-\frac{1}{2}}. \tag{2.2f}$$

The geometric curves of intersection between the interface and the plate are called *contact lines*. As shown in figure 1, these are located at $\hat{z} = \hat{z}_R(\hat{x}, \hat{t})$ and $\hat{z} = \hat{z}_L(\hat{x}, \hat{t})$. These positions are *a priori* unknown. (This is a free-boundary problem.) In order to complete the formulation of the flow problem, it is necessary to pose conditions on the motion of these lines.

The first statement is the *condition of contact*: there is a line along which the liquid thickness is zero,

$$\hat{\eta} = 0 \quad \text{at} \quad \hat{z} = \hat{z}_L, \hat{z}_R \quad \text{for all } \hat{x}, \hat{t}. \tag{2.3a}$$

The second statement concerns the *contact angle*; the slope of the interface at the contact line in the direction normal to the contact line is the tangent of the angle,

$$\nabla\hat{\eta} \cdot \mathbf{\nu} = \pm \tan\alpha, \tag{2.3b}$$

where $\mathbf{\nu}$ is the unit outward vector *on the solid* normal to the contact line and α is called the contact angle. The $\pm$ refers to the pair of contact lines in question; respectively for $+$ and $-$ values $\hat{z} = \hat{z}_L$ and $\hat{z}_R$.

Before these two conditions can be converted into usable boundary conditions, they must be augmented by an *ansatz* that distinguishes one set of materials from another. The ansatz is ultimately dependent upon experimental observation. Among the possibilities are the following:

(i) *Fixed contact line*. The contact line does not move, its position remaining invariant for all time. Hence, $\hat{z}_L$ and $\hat{z}_R$ are time independent.

(ii) *Fixed contact angle*. The contact angle α does not differ from its static (thermodynamic) contact angle α_0 for all time. Here, it is presumed that α_0 is unique though this is a reasonable assumption in only few cases. See Dussan V. (1979) for a discussion.

(iii) *Smooth contact-angle variation*. The contact angle α depends smoothly on the variables of the motion. For example, $\alpha = G(U_{CL})$ where U_{CL} is the speed along the plate of the contact line normal to itself and $G'(U_{CL})$ exists always. The smoothness here *excludes* contact-angle hysteresis.

(iv) *Contact-angle hysteresis*. The contact angle depends on the motion but also on the history of the motion. For example, $\alpha = G(U_{CL})$ is discontinuous at $U_{CL} = 0$ as shown in figure 2. This case is considered common. See Dussan V. (1979) for a discussion.

Finally, there is a boundary condition on the wetted solid. For the case (i) of a fixed contact line, there is the classical no-slip condition:

$$\hat{u}_i = 0 \quad \text{on} \quad \hat{y} = 0, \quad \hat{z}_L \leqslant \hat{z} \leqslant \hat{z}_R. \tag{2.4}$$

However, when the contact line can move as it would in cases (ii), (iii) and (iv), Dussan V. & Davis (1974) have shown on the basis of the kinematics that a non-integrable singularity at the contact line exists as long as the no-slip condition is enforced. Hence, we allow *effective slip* near each contact line. We shall use the following model (e.g. see Greenspan 1978):

$$\hat{u} = \frac{\hat{\kappa}}{\hat{\eta}}\hat{u}_{\hat{y}}, \quad \hat{v} = 0, \quad \hat{w} = \frac{\hat{\kappa}}{\hat{\eta}}\hat{w}_{\hat{y}}. \tag{2.5}$$

Here, the liquid slips over the solid at a speed that is directly proportional to the shear; the magnitude of slip is a numerically small constant $\hat{\kappa}$ divided by the liquid thickness and is thus appreciable only near the contact line.

The analysis of long waves on flat rivulets is a type of lubrication theory. Hence, we shall replace the kinematic condition (2.2a) by an integral relation. This relation is obtained by integrating the continuity condition (2.1b) from $\hat{y} = 0$ to $\hat{y} = \hat{\eta}$ to obtain $\hat{v}$. This form of $\hat{v}$ is substituted into the kinematic condition (2.2a) to obtain

$$\frac{\partial}{\partial \hat{z}}\int_0^{\hat{\eta}} \hat{w}\, d\hat{y} + \hat{\eta}_{\hat{t}} + \frac{\partial}{\partial \hat{x}}\int_0^{\hat{\eta}} \hat{u}\, d\hat{y} = 0.$$

This relation is then integrated from $\hat{z} = \hat{z}_L$ to $\hat{z} = \hat{z}_R$. The result can be put in the following form:

$$\int_{\hat{z}_L}^{\hat{z}_R} \hat{\eta}_t\, d\hat{z} + \int_{\hat{z}_L}^{\hat{z}_R} \frac{\partial}{\partial \hat{x}}\int_0^{\hat{\eta}} \hat{u}\, d\hat{y}\, d\hat{z} = 0, \quad \hat{y} = \hat{\eta}. \tag{2.6}$$

3. The basic state

One solution for rivulets on a long, smooth plate consists of a steady, unidirectional, fully developed flow down the plate in the $\hat{x}$ direction. The fluid wets the solid on a strip of constant width,

$$-L \leqslant \hat{z} \leqslant L. \tag{3.1}$$

The corresponding interfacial shape is likewise $\hat{x}$ independent and so forms a cylindrical meniscus. The flow is driven by the component $g \sin\beta$ of the gravitational acceleration along the plate and is hence determined by a balance between $g\sin\beta$ and the viscous forces. The cylindrical meniscus has a shape determined by surface tension and the component $g\cos\beta$ of the gravitational acceleration normal to the plate. The general forms of flow field and meniscus have been given by Towell & Rothfeld (1966) for the case where the no-slip condition holds on the solid. However, we allow slip on the solid–liquid interface. We wish to consider a special case of this system corresponding to the flow down a *vertical plate*, $\beta = \frac{1}{2}\pi$. The meniscus cross-section then is the arc of a circle. The maximum height of the meniscus is called h_0.

Let us scale all variables in the problem as follows:

$$\text{length} \to h_0, \quad \text{speed} \to U_s = \rho g h_0^2/\mu, \quad \text{pressure} \to \mu U_s/h_0, \quad \text{time} \to h_0/U_s. \tag{3.2}$$

In terms of the above scales, the meniscus $y = H(z)$ satisfies

$$H_{zz}(1+H_z^2)^{-\frac{3}{2}} = -\pi, \quad H(0) = 1, \quad H_z(0) = 0, \tag{3.3a,b,c}$$

$$H(\delta^{-1}) = 0, \quad H_z(\delta^{-1}) = -\tan\alpha_0. \tag{3.3d,e}$$

In the above, π is the constant curvature of the meniscus which corresponds to the pressure excess $\overline{P}$ over the ambient gas pressure. The meniscus is symmetric in z so that only the half-interval $0 \leqslant z \leqslant \delta^{-1}$ is considered,

$$\delta = h_0/L. \tag{3.4}$$

Condition (3.3*b*) reflects the length scale chosen, while condition (3.3*c*) reflects the z symmetry of the meniscus. Condition (3.3*d*) is that of contact while condition (3.3*e*) defines the contact angle.

In terms of the scales (3.2), the basic flow satisfies

$$\overline{U}_{yy} + \overline{U}_{zz} = -1, \tag{3.5a}$$

$$\overline{U}(0,z) = \frac{\kappa}{H(z)}\overline{U}_y(0,z), \quad 0 \leqslant z \leqslant \delta^{-1}, \tag{3.5b}$$

$$\overline{U}_N(H(z),z) = 0, \quad 0 \leqslant z \leqslant \delta^{-1}, \tag{3.5c}$$

where

$$\kappa = \hat{\kappa}/h_0^2. \tag{3.5d}$$

The basic flow $\overline{U}$ is likewise symmetric in z. Condition (3.5*b*) reflects the slip condition on the solid–liquid interface. This effective slip is introduced in the model so that we can describe instabilities of this basic state involving moving contact lines. For (3.5*c*) is the condition of zero shear stress along the x direction. Here the subscript N denotes the normal derivative in the direction $\mathbf{N}$, where $\mathbf{N}$ is the unit outward normal to the interface:

$$\mathbf{N} = (0, 1, -H_z)\, N^{-1} \tag{3.6a}$$

where

$$N = (1 + H_z^2)^{\frac{1}{2}}. \tag{3.6b}$$

The corresponding unit tangent vectors to the undisturbed interface will be needed later. These are

$$\mathbf{T}^{(1)} = (0, H_z, 1)\, N^{-1} \tag{3.6c}$$

and

$$\mathbf{T}^{(2)} = (1, 0, 0). \tag{3.6d}$$

The basic state pressure field $\overline{P}$ within the rivulet represents the pressure excess determined by the curvature of the interface times the surface tension. Since the curvature is constant, so is the pressure:

$$\overline{P} = B_h^{-1} \tag{3.7a}$$

where the *Bond number* B_h based on length scale h_0 is given by

$$B_h = \rho g h_0^2/T, \tag{3.7b}$$

which is the equivalent capillary number $U_s\mu/T$ for this problem.

We shall shortly need to define the Bond number B_L based on length scale L,

$$B_L = \rho g L^2/T. \tag{3.8}$$

4. The basic state for flat rivulets

We shall consider now approximations to $H(z)$ and $\overline{U}(y,z)$ appropriate to flat rivulets, $\delta \ll 1$, having small contact angles. In order to do so, we introduce a new variable,

$$Z = \delta z \tag{4.1}$$

which allows the variations in z to be scaled on L rather than on h_0.

The meniscus equation (3.3a) then takes the form

$$\delta^2 H_{ZZ}(1+\delta^2 H_Z^2)^{-\frac{3}{2}} = -\pi. \tag{4.2}$$

We seek approximate solutions to equation (4.2) by writing

$$H \sim H^{(0)} + O(\delta^2) \tag{4.3a}$$

and

$$\pi \sim \delta^2[\pi^{(0)} + O(\delta^2)], \tag{4.3b}$$

$$\alpha_0 \sim \delta[\alpha^{(0)} + O(\delta^2)], \tag{4.3c}$$

and we assume that $B_L^{-1} = O(1)$ as $\delta \to 0$. This requirement is needed later to retain the surface-tension effects in the stability analysis and will be discussed in § 7. The orderings in expansions (4.3b, c) reflect the fact that a flat rivulet has a small pressure jump across its interface and a small contact angle.

If we substitute forms (4.3) into (4.2) and boundary condition (3.3b, c, d, e), we obtain at order one,

$$H_{ZZ}^{(0)} = -\pi^{(0)}, \quad H^{(0)}(0) = 1, \quad H_Z^{(0)}(0) = 0, \tag{4.4a, b, c}$$

$$H^{(0)}(1) = 0, \quad H_z^{(0)}(1) = -\alpha^{(0)}. \tag{4.4, d, e}$$

The solution of (4.4) is given by

$$H^{(0)} = 1 - Z^2, \quad \pi^{(0)} = 2, \quad \alpha^{(0)} = 2. \tag{4.5a, b, c}$$

Clearly, approximations to any order can be obtained in this way.

The flow equation (3.5) can be approximated in the same way as the meniscus solution. We introduce Z by equation (4.1) and write

$$Y = y. \tag{4.6a}$$

We now let

$$\overline{U} = \overline{U}^{(0)} + O(\delta^2). \tag{4.6b}$$

By substituting from (4.6) into system (3.5), we obtain at $O(1)$ the following system:

$$\overline{U}_{YY}^{(0)} = -1, \quad \overline{U}^{(0)}(0,Z) = \frac{\kappa}{H^{(0)}}\overline{U}_Y^{(0)}(0,Z), \quad \overline{U}_Y^{(0)}(H^{(0)},Z) = 0. \tag{4.7a, b, c}$$

The solution to system (4.7) is

$$\overline{U}^{(0)} = H^{(0)}Y - \tfrac{1}{2}Y^2 + \kappa. \tag{4.8}$$

We see that the effect of slip on the basic state is only minor. The slip induces at leading order in δ a small translation onto $\overline{U}^{(0)}$ but leaves the velocity gradients unchanged. Clearly, higher-order approximations may be obtained in this way.

It is worth noting that the limit $\delta \to 0$ is *formally* a singular perturbation of (3.5a)

since in the limit the derivative $\partial^2/\partial Z^2$ is lost. However, the approximate solution (4.8) does satisfactorily satisfy the full problem with error $O(\delta^2)$ *uniformly*. Hence, the domain $-1 \leqslant Z \leqslant 1$, $0 \leqslant Y \leqslant H(Z)$ is such that a uniformly valid approximate solution is obtained by using the 'outer' solution.

5. Disturbance equations

Let us allow disturbances of the basic state as follows:

$$\mathbf{v} = (\bar{U}(y,z),0,0)+\epsilon\mathbf{v}', \quad p = \bar{P}+\epsilon p', \quad h = H(z)+\epsilon h'. \qquad (5.1\,a,b,c)$$

The contact-line positions and contact angles are given by

$$(z_L, z_R) = (-\delta^{-1}, \delta^{-1})+\epsilon(-z_L', z_R') \qquad (5.1\,d)$$

and

$$\alpha_{R,L} = \alpha_0+\epsilon\alpha_{R,L}'. \qquad (5.1\,e)$$

If forms (5.1) are substituted into the non-dimensional form of (2.1)–(2.4), and linearized in disturbance quantities, the linearized-disturbance system results at order ϵ. If the primes are dropped, this system is as follows:

$$R[v_{i_t}+\bar{U}v_{i_x}+(\bar{U}_y v+\bar{U}_z w)\,T_i^{(2)}] = \sigma_{ij,j}, \qquad (5.2\,a)$$

$$v_{i,i} = 0, \qquad (5.2\,b)$$

$$\sigma_{ij}N_jN_i = B_h^{-1}K \quad \text{on} \quad y = H, \qquad (5.2\,c)$$

$$\sigma_{ij}N_jT_i^{(1)} = (\partial\bar{U}/\partial T^{(1)})\,N^{-1}h_x \quad \text{on} \quad y = H, \qquad (5.2\,d)$$

$$\sigma_{ij}N_jT_i^{(2)} = (\partial\bar{U}/\partial T^{(1)})\,N^{-2}h_z-\frac{\partial}{\partial y}(\partial\bar{U}/\partial N)\,h \quad \text{on} \quad y = H, \qquad (5.2\,e)$$

$$v = h_t+\bar{U}h_x+H_z w \quad \text{on} \quad y = H, \qquad (5.2\,f)$$

$$\sigma_{ij} = -p\delta_{ij}+v_{i,j}+v_{j,i}, \qquad (5.2\,g)$$

$$K = (N^{-3}h_z)_z+N^{-1}h_{xx}, \qquad (5.2\,h)$$

and the Reynolds number R is given by

$$R = U_s h_0/\nu. \qquad (5.2\,i)$$

In the course of deriving system (5.2), we have transferred the interface from its exact position to its undisturbed position $y = H$ and have used the basic-state solutions for simplification.

The above system must be augmented by conditions at the contact lines plus conditions on the solid–liquid interface. The latter conditions involve effective slip which is utilized when the contact lines move. This slip takes the form

$$v = 0, \quad y = 0 \qquad (5.3\,a)$$

and

$$u = \frac{\kappa}{H}\left[u_y-\frac{h}{H}\bar{U}_y\right], \quad y = 0, \qquad (5.3\,b)$$

$$w = \frac{\kappa}{H}w_y, \quad y = 0. \qquad (5.3\,c)$$

Form (5.3*a*) implies that no liquid penetrates the solid while forms (5.3*b*, *c*) give the linearized slip conditions. The contact-line conditions are formulated in § 6.

The linearized form of the volume-flow condition (2.6) replaces the kinematic condition and is given as follows:

$$\int_{-\delta^{-1}}^{\delta^{-1}} h_t dz + \int_{-\delta^{-1}}^{\delta^{-1}} \left[\bar{U}(H,z)\, h_x + \int_0^H u_x dy \right] dz = 0. \tag{5.3d}$$

6. Contact-line conditions

The conditions to be applied at the contact lines depend on the presence or absence of contact-line motion. We consider the same possibilities as Davis (1980), namely (i) fixed contact lines, (ii) fixed contact angle (including effective slip) and (iii) smooth contact-angle variations (including effective slip). Cases (ii) and (iii) involve moving contact lines, the lines no longer being straight.

The contact-line conditions are obtained from the condition of contact, (2.3*a*), and the contact-angle condition, (2.3*b*).

Let us consider the contact line at $z = \delta^{-1} + \epsilon z_R(x,t)$. The linearized form of (2.3*a*) is

$$h(x, \delta^{-1}, t) = -H_z(\delta^{-1})\, z_R(x,t). \tag{6.1a}$$

Likewise,

$$h(x, -\delta^{-1}, t) = H_z(-\delta^{-1})\, z_L(x,t). \tag{6.1b}$$

The normal vector $\boldsymbol{\nu}$ to the contact line defined in (2.3*b*) is

$$\boldsymbol{\nu} = (-\epsilon z_{R_x}, 0, 1)(1+\epsilon^2 z_{R_x}^2)^{-\frac{1}{2}}$$

so that the linearized form of (2.3*b*) is

$$h_z(x, \delta^{-1}, t) = -H_{zz}(\delta^{-1})\, z_R - \alpha_R \sec^2 \alpha_0. \tag{6.2a}$$

Likewise,

$$h_z(x, -\delta^{-1}, t) = H_{zz}(-\delta^{-1})\, z_L + \alpha_L \sec^2 \alpha_0. \tag{6.2b}$$

We can now consider cases of contact-line boundary conditions.

Fixed contact lines. When the contact lines are fixed in space at their basic state positions, $z_R = z_L \equiv 0$, so that equations (6.1) give

$$h = 0, \quad z = \pm \delta^{-1}. \tag{6.3}$$

Since the contact lines are stationary, the no-slip condition on the solid–liquid interface is adequate. Hence, $\kappa = 0$ in forms (5.3) and

$$\mathbf{v} = 0 \quad \text{on} \quad y = 0, \quad |z| \leqslant \delta^{-1}. \tag{6.4}$$

Fixed contact angles. When the contact angles are fixed for all time, $\alpha = \alpha_0$ at each line and hence $\alpha_R = \alpha_L \equiv 0$. Since $z_R, z_L \neq 0$, we can eliminate those between (6.1) and (6.2). We then obtain

$$h_z - \frac{H_{zz}}{H_z} h = 0, \quad z = \pm \delta^{-1}. \tag{6.5}$$

Smooth contact-angle variation. If it is assumed that the instantaneous contact angle depends smoothly on the contact-line speed (in the direction normal to the contact line), then, near $z = \delta^{-1}$,

$$\alpha_0 + \epsilon \alpha_R = G(0 + \epsilon U_{CL}). \tag{6.6}$$

The linearized version of (6.6) is

$$\alpha_R = G_1 U_{CL},$$

where

$$G_1 \equiv G'(0),$$

and

$$U_{CL} = \mathbf{v}\cdot\mathbf{\nu}_R]_{z=\delta^{-1}+\epsilon z_R}$$
$$= w - \bar{U} z_{R_x} + O(\epsilon), \quad z = \delta^{-1}.$$

However, if we use the kinematic condition (5.2*f*) and form (6.1*a*) to eliminate both w and z_R, we have

$$U_{CL} = -\frac{1}{H_z} h_t, \quad z = \delta^{-1}.$$

Hence,

$$\alpha_R = -\frac{G_1}{H_z} h_t, \quad z = \delta^{-1}. \tag{6.7}$$

When result (6.7) is compared with form (6.2*a*) we find that

$$h_z - \frac{H_{zz}}{H_z} h = G_1 \frac{(1+H_z^2)}{H_z} h_t, \quad z = \pm\delta^{-1}. \tag{6.8}$$

Here we have used the relation that $\tan\alpha_0 = \mp H_z$ at $z = \pm\delta^{-1}$.

7. Disturbance equations for long waves on rivulets

We wish to consider the stability of rivulet flows in the case that the maximum depth h_0 is much shorter than the wavelength λ of the disturbances. If we define a non-dimensional wavenumber, $k = 2\pi h_0/\lambda$, we examine the case $k \ll 1$ for R and δ held fixed but arbitrary.

We rescale the variables as follows:

$$\left.\begin{array}{l} T = kt, \quad X = kx, \quad Y = y, \quad Z = \delta z, \\ U = u, \quad V = v/k, \quad W = \delta w/k, \quad P = kp, \quad h = h. \end{array}\right\} \tag{7.1}$$

In the scalings (7.1) we have used facts appropriate to long waves on film flow. Among these are that the complex growth rate is $O(k)$ so that kt scales the time. Furthermore, we allow $p = O(1/k)$, as is appropriate in lubrication theory, though it turns out in all cases that the leading term in pressure is zero so that $p = O(1)$. The scalings in δ are taken so that, as $\delta \to 0$, the full equation of continuity is preserved.

We introduce normal modes for each dependent variable ϕ as follows:

$$\phi(X, Y, Z, T) = \Phi(Y, Z)\, e^{i(X-cT)}. \tag{7.2}$$

We can then see that the complex eigenvalue, $c = c_R + ic_I$, determines the stability of the basic state. If forms (7.1) and (7.2) are substituted into (5.2), (5.3), and (6.3), (6.5) and (6.8), we obtain the scaled, linearized disturbance system in terms of normal modes. These are as follows:

$$kR\{i(\bar{U}-c)\,U + \bar{U}_Y\, V + \bar{U}_Z\, W\} = -iP + k^2 U + U_{YY} + \delta^2 U_{ZZ}, \tag{7.3a}$$

$$ik^3 R(\bar{U}-c)\, V = -P_Y + k^2\{k^2 V + V_{YY} + \delta^2 V_{ZZ}\}, \tag{7.3b}$$

$$ik^3 R(\bar{U}-c)\, W = -\delta^2 P_Z + k^2\{k^2 W + W_{YY} + \delta^2 W_{ZZ}\}, \tag{7.3c}$$

$$iU + V_Y + W_Z = 0, \tag{7.3d}$$

$$-P+2k^2[V_Y+\delta^2 H_Z^2 W_Z - H_Z(W_Y+\delta^2 V_Z)]N^{-2} = kB_h^{-1}[\delta^2(N^{-3}h_Z)_Z - k^2N^{-1}h], \quad Y=H, \tag{7.3e}$$

$$-i\delta^2[\overline{U}_Z + H_Z\overline{U}_Y]h + (1-\delta^2H_z^2)[W_Y+\delta^2V_Z] + 2\delta^2H_Z(V_Y-W_Z) = 0, \quad Y=H, \tag{7.3f}$$

$$U_Y + ik^2V - H_Z[\delta^2U_Z + ik^2W] - \delta^2\overline{U}_Z h_Z + [\overline{U}_Y - \delta^2H_Z\overline{U}_Z]_Y h = 0, \quad Y=H, \tag{7.3g}$$

$$\int_{-1}^{1}[\overline{U}(H,Z)-c]\,h\,dZ + \int_{-1}^{1}\int_0^H U\,dY\,dZ = 0, \tag{7.3h}$$

where

$$N = (1+\delta^2H_z^2)^{\frac{1}{2}} \tag{7.3i}$$

and we have used (3.5c) to simplify (7.3g).

The boundary condition on the solid–liquid interface is given by (5.3). These are

$$V = 0, \qquad Y = 0, \tag{7.4a}$$

$$U = \frac{\kappa}{H}\left[U_Y - \frac{h}{H}\overline{U}_Y\right], \quad Y=0, \tag{7.4b}$$

$$W = \frac{\kappa}{H}W_Y, \qquad Y=0. \tag{7.4c}$$

At the contact lines, we have three distinct cases.

When the *contact lines are fixed*, $\kappa = 0$ in (7.4b,c) and (6.3) gives

$$h = 0, \quad Z = \pm 1. \tag{7.4d}$$

When the *contact angles are fixed*, $\kappa \neq 0$ in (7.4b,c) and (6.5) gives

$$h_Z - \frac{H_{ZZ}}{H_Z}h = 0, \quad Z = \pm 1. \tag{7.4e}$$

When there is *smooth contact-angle variation*, $\kappa \neq 0$ in (7.4b,c) and (6.8) gives

$$\delta^2\left(h_Z - \frac{H_{ZZ}}{H_Z}h\right) = -\frac{ikcG_1N^2}{H_Z}h, \quad Z=\pm 1. \tag{7.4f}$$

We now exploit the fact that k is small and express all dependent variables ϕ and complex eigenvalue c in powers of k,

$$\phi = \phi_0 + k\phi_1 + O(k^2) \tag{7.5a}$$

and

$$c = c_0 + kc_1 + O(k^2). \tag{7.5b}$$

At order unity in k, the system (7.3) and (7.4) has the form

$$U_{0YY} + \delta^2 U_{0ZZ} - iP_0 = 0, \tag{7.6a}$$

$$P_{0Y} = 0, \tag{7.6b}$$

$$\delta^2 P_{0Z} = 0, \tag{7.6c}$$

$$iU_0 + V_{0Y} + W_{0Z} = 0, \tag{7.6d}$$

$$P_0 = 0, \quad Y=H, \tag{7.6e}$$

$$-i\delta^2[\overline{U}_Z + H_Z\overline{U}_Y]h_0 + (1-\delta^2H_z^2)[W_{0Y}+\delta^2V_{0Z}] + 2\delta^2H_Z(V_{0Y}-W_{0Z}) = 0, \quad Y=H, \tag{7.6f}$$

$$U_{0Y} - \delta^2H_ZU_{0Z} - \delta^2\overline{U}_Z h_{0Z} + [\overline{U}_Y - \delta^2H_Z\overline{U}_Z]_Y h_0 = 0, \quad Y=H, \tag{7.6g}$$

$$\int_{-1}^{1}[\bar{U}(H,Z)-c_0]h_0\,dZ+\int_{-1}^{1}\int_0^H U_0\,dY\,dZ=0, \tag{7.6h}$$

$$V_0=0, \quad Y=0, \tag{7.6i}$$

$$U_0=\frac{\kappa}{H}\left[U_{0Y}-\frac{\bar{U}_Y}{H}h_0\right], \quad Y=0, \tag{7.6j}$$

$$W_0=\frac{\kappa}{H}W_{0Y}, \quad Y=0. \tag{7.6k}$$

The contact-line conditions are: (i) for fixed contact lines, $\kappa=0$ and

$$h_0=0, \quad Z=\pm 1; \tag{7.7a}$$

(ii) for fixed contact angles, $\kappa \neq 0$ and

$$h_{0Z}-\frac{H_{ZZ}}{H_Z}h_0=0, \quad Z=\pm 1; \tag{7.7b}$$

and (iii) for smooth contact-angle variation $\kappa \neq 0$ and

$$h_{0Z}-\frac{H_{ZZ}}{H_Z}h_0=0, \quad Z=\pm 1. \tag{7.7c}$$

At order k, the system (7.3) and (7.4) has the form

$$U_{1YY}+\delta^2 U_{1ZZ}-iP_1=R\{i(\bar{U}-c_0)U_0+\bar{U}_Y V_0+\bar{U}_Z W_0\}, \tag{7.8a}$$

$$P_{1Y}=0, \tag{7.8b}$$

$$\delta^2 P_{1Z}=0, \tag{7.8c}$$

$$iU_1+V_{1Y}+W_{1Z}=0, \tag{7.8d}$$

$$-P_1=B_L^{-1}(N^{-3}h_{0Z})_Z-B_\lambda^{-1}N^{-1}h_0, \quad Y=H, \tag{7.8e}$$

$$-\delta^2[\bar{U}_Z+H_Z\bar{U}_Y]h_1+(1-\delta^2H_Z^2)[W_{1Y}+\delta^2V_{1Z}]+2\delta^2H_Z(V_{1Y}-W_{1Z})=0, \quad Y=H, \tag{7.8f}$$

$$U_{1Y}-\delta^2H_ZU_{1Z}-\delta^2\bar{U}_Zh_{1Z}+[\bar{U}_Y-\delta^2H_Z\bar{U}_Z]_Y h_1=0, \quad Y=H, \tag{7.8g}$$

$$\int_{-1}^{1}\{[\bar{U}(H,Z)-c_0]h_1-c_1h_0\}\,dZ+\int_{-1}^{1}\int_0^H U_1\,dY\,dZ=0, \tag{7.8h}$$

$$V_1=0, \quad Y=0, \tag{7.8i}$$

$$U_1=\frac{\kappa}{H}\left[U_{1Y}-\frac{\bar{U}_Y}{H}h_1\right], \quad Y=0, \tag{7.8j}$$

$$W_1=\frac{\kappa}{H}W_{1Y}, \quad Y=0. \tag{7.8k}$$

In addition

$$h_1=0, \quad Z=\pm 1, \tag{7.9a}$$

$$h_{1Z}-\frac{H_{ZZ}}{H_Z}h_1=0, \quad Z=\pm 1, \tag{7.9b}$$

$$h_{1Z}-\frac{H_{ZZ}}{H_Z}h_1+ic_0g_1h_0/H_Z=0, \quad Z=\pm 1. \tag{7.9c}$$

In equation (7.8 *e*) we have written

$$B_\lambda = B_h k^{-2}, \tag{7.10}$$

where B_λ is the Bond number based on length scale λ. In equation (7.9*c*)

$$g_1 \equiv G_1 \delta^{-2} \tag{7.11}$$

is assumed to be $O(1)$ as $\delta \to 0$ in order to retain the effect of contact-angle variations during contact-line motion.

In the case of film flow, treated in § 8, the term involving B_L is absent so that in order to retain the effects of surface tension we must make the standard assumption that $B_\lambda = O(1)$ as $k \to 0$. However, in rivulet flows, as long as $k^2/\delta^2 \ll 1$, B_λ^{-1} is negligible and the effect of surface tension is effectively much larger.

We shall need only some of the order-k^2 equations. These are as follows:

$$P_{2Y} = V_{0YY} + \delta^2 V_{0ZZ}, \tag{7.12a}$$

$$\delta^2 P_{2Z} = W_{0YY} + \delta^2 W_{0ZZ}, \tag{7.12b}$$

$$-P_2 + 2[V_{0Y} + \delta^2 H_Z^2 W_{0Z} - H_Z(W_{0Y} + \delta^2 V_{0Z}]\, N^{-2} = B_L^{-1}(N^{-3}h_{1Z})_Z - B_\lambda^{-1} N^{-1} h_1, \quad Y = H. \tag{7.12c}$$

8. Film flow

Film flow involves a continuous layer of infinite extent in the z direction. The stability of two-dimensional waves on such a vertical film was first determined using long-wave expansions by Yih (1963). Our present analysis reproduces these results.

The film-flow equations are obtained from system (7.3) by writing

$$H(Z) \equiv 1, \quad \bar{U}(Y,Z) = Y - \tfrac{1}{2}Y^2 + \kappa, \quad W \equiv 0, \tag{8.1a, b, c}$$

and seeking z-independent solutions. We shall call this the film-flow limit.

The solutions at order unity satisfy the system (7.6) specialized by the film-flow limit. These are

$$U_0 = Y, \quad V_0 = -\tfrac{1}{2} i Y^2, \quad P_0 = 0, \quad h_0 = 1. \tag{8.2a, b, c, d}$$

Note that since $P_0 = 0$ the unscaled pressure field p of (7.2) satisfies $p = O(1)$.

The eigenvalue c_0 is obtained through substitution into form (7.6*h*). We find that

$$c_0 = 1 + \kappa. \tag{8.3}$$

The solutions at order k satisfy the system (7.8) specialized by the film-flow limit. These are

$$U_1 = i\{\tfrac{1}{2}B_\lambda^{-1}Y^2 + \tfrac{1}{6}R(\tfrac{1}{4}Y^4 - Y^3) + A_1 Y + \kappa(\tfrac{1}{3}R - B_\lambda^{-1})\}, \tag{8.4a}$$

$$V_1 = \tfrac{1}{6}B_\lambda^{-1}Y^3 + \tfrac{1}{24}R(\tfrac{1}{5}Y^5 - Y^4) + \tfrac{1}{2}A_1 Y^2 + \kappa(\tfrac{1}{3}R - B_\lambda^{-1})\, Y, \tag{8.4b}$$

$$P_1 = B_\lambda^{-1}, \quad h_1 = i\{B_\lambda^{-1} - \tfrac{1}{3}R + A_1\}, \tag{8.4c, d}$$

where we have chosen the arbitrary multiplication constant of linear theory by selecting P_1. Here the arbitrary constant A_1 comes from the complementary solution of the order-k equations and as such reproduces the order-unity solutions given in (8.2). Without loss of generality, we can set

$$A_1 = 0. \tag{8.5}$$

Using the forms of U_1 and h_1 given in (8.4*a, d*), we substitute into (7.9*h*) to obtain

$$c_1 = i\{\tfrac{2}{15}R - \tfrac{1}{3}B_\lambda^{-1} + \kappa(\tfrac{1}{3}R - B_\lambda^{-1})\}. \tag{8.6}$$

We thus see from (8.3) and (8.6) for film flow on a vertical plate that

$$c \sim 1 + \kappa + ik\{\tfrac{2}{15}R - \tfrac{1}{3}B_\lambda^{-1} + \kappa(\tfrac{1}{3}R - B_\lambda^{-1})\} + O(k^2) \tag{8.7}$$

so that small-amplitude long-wave disturbances grow when

$$R > R_C = \tfrac{5}{2}B_\lambda^{-1}\left[\frac{1-3\kappa}{1+\frac{5}{2}\kappa}\right]. \tag{8.8}$$

This is precisely the result of Yih (1963) when there is no slip ($\kappa = 0$) at the liquid–solid boundary. Since κ is numerically very small, condition (8.8) essentially has $R_C = \tfrac{5}{2}B_\lambda^{-1}$. The slippage merely destabilizes the film by a minor amount.

In deriving result (8.8) we have taken the Bond number based on wavelength $B_\lambda = O(1)$ as $k \to 0$, which is the standard assumption invoked for incorporating the effects of surface tension into the dynamic stability criterion. Thus, surface tension on the interface is able to delay the onset of instability since, in two-dimensional flow, the capillary pressure gradients aways tend to oppose interfacial corrugations in the sense that bulk fluid is pumped from the thick region of the layer into the thin.

9. Solutions for small δ for fixed contact lines

In this section we take the contact lines to be fixed so that $h = 0$ at $Z = \pm 1$ according to (7.4*d*). We then need not allow slip at the solid so we take $\kappa = 0$ in (7.4*b, c*).

We see from (7.6*b*) and (7.6*c*) that P_0 is constant while from condition (7.6*e*) we have

$$P_0 \equiv 0. \tag{9.1}$$

The streamwise velocity U_0 is then obtained from (7.6*a*) subject to conditions (7.6*g*) and (7.6*j*). We find for small δ that

$$U_0 = h_0(Z)\, Y + O(\delta^2). \tag{9.2}$$

Equations (7.8*b*) and (7.8*c*) show that P_1 is constant. We take, say,

$$P_1 = 2B_L^{-1}. \tag{9.3}$$

The leading-order boundary perturbation h_0 is determined through (7.8*e*) and conditions (7.7*a*). The solution for small δ is

$$h_0 = 1 - Z^2. \tag{9.4}$$

In obtaining form (9.4) we have neglected $B_\lambda^{-1}B_L = k^2/\delta^2$ in (7.9*e*) as discussed in §7.

The cross-speed W_0 is determined by (7.12*b*) and conditions (7.6*f*) and (7.6*k*). We find that

$$W_0 = O(\delta^2). \tag{9.5}$$

We then determine form V_0 from (7.6*d*), condition (7.6*i*) and form (9.2) as follows:

$$V_0 = -\tfrac{1}{2}ih_0(Z)\, Y^2 + O(\delta^2). \tag{9.6}$$

Given the forms (9.2) and (9.4), we can compute c_0 from (7.6*h*). For small δ we find that

$$c_0 = 0{\cdot}686 + O(\delta^2). \tag{9.7}$$

We now turn to solutions of the order-k corrections.

The correction to the streamwise velocity is determined by (7.9a) and the condition (7.8j). If we use the value of P_1 given by (9.3), then we find that

$$U_1 = i\{B_L^{-1}Y^2 + \tfrac{1}{6}Rh_0(\tfrac{1}{4}HY^4 - c_0Y^3) + A_1(Z)\,Y\} + O(\delta^2), \tag{9.8}$$

which satisfies the no-slip condition at $Y = 0$.

From the form (9.8) we can obtain h_1 through (7.8g). We find that

$$h_1 = i\{2B_L^{-1}H + \tfrac{1}{2}Rh_0[\tfrac{1}{3}H^4 - c_0H^4] + A_1(Z)\} + O(\delta^2). \tag{9.9}$$

The function $A_1(Z)$ in results (8.7) and (9.9) is seen to involve only terms that reproduce the complementary solutions U_0 and h_0 in U_1 and h_1, respectively. Hence, the forms that satisfy the boundary conditions have $A_1(Z) = a_1h_0(Z)$, where a_1 is an arbitrary constant. We can then choose $a_1 = 0$ and still have h_1 of (7.8) satisfy the end conditions $h_1(\pm 1) = 0$.

If we substitute the forms for U_1, h_0 and h_1 into (7.9h) and use $A_1(Z) = 0$, we obtain

$$c_1 = 0{\cdot}031[R - 14{\cdot}8B_L^{-1}]\,i. \tag{9.10}$$

Hence, flat rivulets on vertical walls are stable to small long-wave disturbances as long as

$$R < R_C = 14{\cdot}8B_L^{-1}. \tag{9.11}$$

This region of stability is much greater than that of film flow since we are presuming that $k \ll \delta$. The immobility of the contact lines is responsible for this stabilization as will be seen in § 10.

In the present case of rivulets with fixed contact lines, the surface tension on the interface is seen to delay the onset of instability in much the same way as in film flow. Even though the rivulet is three-dimensional, capillary pressures oppose interfacial corrugations.

10. Solutions for small δ for fixed contact angles

In this section we take the contact angles to be fixed. Hence, the contact lines are free to move and we allow effective slip on the solid–liquid interface.

We see from equations (7.6b) and (7.6c) that P_0 is constant while from condition (7.6e) we have that

$$P_0 \equiv 0. \tag{10.1}$$

The streamwise velocity U_0 is then obtained from (7.6a),

$$U_0 = A_0(Z)\,Y + B_0(Z) + O(\delta^2) \tag{10.2}$$

and the modification h_0 in the boundary position is obtained from (7.6g),

$$h_0 = A_0(Z) + O(\delta^2). \tag{10.3}$$

Equations (7.8b) and (7.8c) show that P_1 is constant. We take, say,

$$P_1 = 2B_L^{-1}. \tag{10.4}$$

We now use relation (7.8e) to evaluate $A_0(Z)$ as follows:

$$A_0(Z) = -(Z^2 + a_0Z + b_0). \tag{10.5}$$

If we use forms (10.3) and (10.5) in the contact-line condition (7.7*b*), we find that

$$b_0 = 1 \tag{10.6a}$$

and

$$a_0 \text{ arbitrary.} \tag{10.6b}$$

Finally, we use the slip condition (7.6*j*) and forms (10.2) and (10.3) to evaluate $B_0(Z)$,

$$B_0(Z) \equiv 0. \tag{10.7}$$

Hence, we have from forms (10.2)–(10.7) that

$$U_0 = h_0(Z)\,Y + O(\delta^2) \tag{10.8a}$$

and

$$h_0 = -(1+Z^2) + a_0 Z + O(\delta^2), \tag{10.8b}$$

where the constant a_0 is arbitrary.

We can now evaluate c_0 using condition (7.6*h*). We find that

$$c_0 = 0{\cdot}457 + \kappa + O(\delta^2). \tag{10.9}$$

The lateral speed W_0 is determined from (7.12*b*), condition (7.6*f*) and slip condition (7.6*k*). We find that

$$W_0 = O(\delta^2). \tag{10.10}$$

Finally, the continuity equation (7.6*d*) and the condition (7.6*i*) give the normal speed V_0,

$$V_0 = -\tfrac{1}{2} i h_0\, Y^2 + O(\delta^2). \tag{10.11}$$

We now turn to the determination of the order-k corrections. The streamwise speed U_1, is given by (7.8*a*), the value $P_1 = 2B_L^{-1}$ and the forms for U_0, W_0 and $\bar{U}$. We find that

$$U_1 = i\{B_L^{-1} Y^2 + \tfrac{1}{6} R h_0 [\tfrac{1}{4} H Y^4 - (c_0 - \kappa)\, Y^3] + A_1(Z)\, Y + B_1(Z)\} + O(\delta^2). \tag{10.12}$$

From (10.12) and (7.8*g*) we have that

$$h_1 = i\{2B_L^{-1} H + \tfrac{1}{2} R h_0 [\tfrac{1}{3} H^4 - (c_0 - \kappa) H^2] + A_1(Z)\} + O(\delta^2). \tag{10.13}$$

But, the slip condition (7.8*j*) gives that

$$B_1 = -\kappa\{2B_L^{-1} + \tfrac{1}{2} R h_0 [\tfrac{1}{3} H^3 - (c_0 - \kappa) H]\} + O(\delta^2). \tag{10.14}$$

We need only calculate $A_1(Z)$ before we can find c_1.

At this point we have *no deductive way* of determining $A_1(Z)$. Hence, we use the following procedure. We shall *pose* that A_1 is a minimal polynomial, determine its coefficients and hence obtain a unique functional form. The method is reminiscent of Benjamin's (1957) use of power series expansion in the independent variable for the determination of c_1 for film flow.

In order to evaluate A_1 we assume the form

$$A_1(Z) = b_1 + a_1 Z + d_1 Z^2 \tag{10.15}$$

and substitute into the contact-line condition (7.9*b*). We find that $d_1 - b_1 = 4B_L^{-1}$ and a_1 arbitrary so that

$$A_1(Z) = a_1(1+Z^2) + b_1 Z + 4B_L^{-1} Z^2. \tag{10.16}$$

Without loss of generality we can take take $a_1 = 0$ since this term only reproduces the complementary solutions. Again, the value of b_1 does not affect the stability condition.

The value of c_1 can be calculated from (7.8*h*) using (10.12)–(10.15). We find that

$$c_1 = \{(0{\cdot}228+\kappa)B_L^{-1}+(0{\cdot}0031+0{\cdot}0491\kappa)R\}i. \tag{10.17}$$

We thus see that for fixed contact angles that vertical flat rivulets are always unstable to small-amplitude long waves.

We note here that surface tension on the interface augments the kinematic long-wave instability. Here, the capillary pressure gradient aids the instability by the same mechanism responsible for the break-up of a cylinder into droplets (Rayleigh 1879).

11. Solutions for small δ for smooth contact-angle variations

In this section we allow the contact line to move freely by allowing effective slip on the solid–liquid interface. The contact angle is assumed to be a smooth function of contact-line speed.

The analysis of the previous section applies here line-for-line until the evaluation of $A_1(Z)$ in (10.16). Since equation (7.9*c*) must replace (7.9*b*), the coefficient $4B_L^{-1}$ in A_1 must be replaced by $4B_L^{-1}-g_1c_0$. As a result, the relevant complex disturbance speeds are as follows:

$$c_0 = 0{\cdot}457+\kappa \tag{11.1}$$

and

$$c_1 = \{(0{\cdot}228+\kappa)B_L^{-1}+(0{\cdot}0031+0{\cdot}0491\kappa)R-(0{\cdot}0261+0{\cdot}0571\kappa)g_1\}i. \tag{11.2}$$

We thus see that the change in contact angle with contact line speed is a stabilizing effect whose magnitude $g_1 = G_1/\delta^2$ may, indeed, be large.

Surface tension on the interface again augments the instability as in the case of fixed contact angles.

12. Conclusions

In the present work we have formulated the linearized stability theory of dynamic rivulets flowing down vertical walls. The three types of contact-line conditions defined by Davis (1980) have been considered: (i) fixed contact lines, (ii) fixed contact angles and (iii) contact angles that vary smoothly with contact-line speeds. In cases (ii) and (iii), since the contact lines are free to move, effective slip between the liquid and solid is posed. In each of these cases, though, the slip coefficient and hence the slip model have only a negligible effect on the final results.

The undisturbed rivulet consists of unidirectional parallel flow down the plate, a cylindrical meniscus of circular cross-section and straight contact lines. Its maximum depth is h_0 and its width is $2L$. The present theory consists of examining travelling-wave disturbances of wavelength λ much greater than the rivulet depth h_0 so that $k = 2\pi h_0/\lambda$ satisfies

$$k \ll 1.$$

Furthermore, the rivulet is considered quite flat so that, if $\delta = h_0/L$, then

$$\delta^2 \ll 1.$$

We presume that $k \ll \delta$ so that arbitrary δ is included in the analysis and the rivulet resembles film flow for small δ. The results of this theory for long waves on flat rivulets depend on contact-line boundary conditions.

When the contact lines are fixed, there is a critical Reynolds number R_C above which the basic rivulet is unstable to small long-wave disturbances. Here

$$R_C = 14{\cdot}8B_L^{-1}.$$

This compares to the film flow critical Reynolds number

$$R_C = 2{\cdot}5B_\lambda^{-1}.$$

Hence, the flat rivulet, having fixed contact lines, is quite a bit more stable than the corresponding film flow. This stabilization results from surface tension being a more effective stabilizer on length scale L, the width, than on the length scale λ and the immobility of the contact lines. It is the curvature in the rivulet cross-section that dominates the stabilization of the rivulet for $k \ll \delta$. The phase speed of the most critical disturbance is c_R. In the film flow case, $c_R \approx 1$, which is twice the surface speed in the basic film flow. In the case of the rivulet with fixed contact lines, $c_R \approx 0{\cdot}686$ where the surface speed in the basic rivulet varies from $\frac{1}{2}$ at the centre-line to zero at the contact line. If one calculates the speed averaged over the rivulet cross-section, one again finds that small-amplitude waves travel at about twice the *average* surface speed.

When the contact lines move with fixed contact angle, there is no critical Reynolds number; the rivulet is always unstable and the waves travel substantially slower than before. The high degree of instability here is apparently associated with the extra degree of freedom available to the rivulet. However, such arguments are not without peril in free-surface problems as shown by Davis & Homsy (1980) is a different context.

When the contact lines move such that the contact angle is a smooth function of contact-line speed, a new parameter G_1 enters the problem. If α is the contact angle and U_{CL} is the contact-line speed, then $G_1 \equiv [d\alpha/dU_{CL}]_{U_{CL}=0}$. The stability analysis shows that the stability condition corresponding to fixed contact angle applies except that the steepening of the contact angle with contact-line motion acts to stabilize. This stabilization is proportional to g_1, where $g_1 \equiv G_1\delta^{-2}$ and g_1 is assumed order unity as $\delta \to 0$. Here, the complex growth rate c for $\delta \to 0$, has the approximate form

$$c \sim 0{\cdot}457 + ik\{0{\cdot}228B_L^{-1} + 0{\cdot}0031R - 0{\cdot}0261g_1\} + O(k^2),$$

where we have neglected terms proportional to the slip coefficient κ. Hence, if g_1 is large enough, the damping effect of changes in contact angle can stabilize an otherwise unconditionally unstable rivulet. (The fixed-contact-angle case is obtained by letting $g_1 \to 0$.) Furthermore, one is tempted to think that the physically realistic case of contact-angle hysteresis might be approached from the present case with $g_1 \to \infty$. This would suggest that contact-angle hysteresis would lead to absolute stability of rivulets against small-amplitude long waves. This interpretation, though, remains at the moment only conjectural.

In any case, the present analysis supports the finding of Davis (1980) that increase in contact angle with contact-line speed is a purely dissipative effect. In his analysis of static rivulets, the stability condition is left unchanged by $g_1 \neq 0$ in the same way that viscosity leaves the threshold wavenumber unchanged in the capillary instability of a Rayleigh jet. In the present analysis of a dynamic rivulet, the effect of dynamically changing contact angles again acts in the direction of damping leading to a stabilization of an otherwise unstable rivulet. It is interesting to think that this dissipation effect might hold in general, i.e. in a broad range of stability problems where finite-amplitude disturbances are present. Such an insight would be extremely helpful in more complicated situations where analysis is not possible. This result is, however, not available at present.

One common feature of the two moving-contact-line cases considered is that surface tension on the interface acts to reinforce the kinematic instability. This is to be expected since the basic rivulet has a curved (cylindrical) meniscus and so it is susceptible to a Rayleigh (1879)-type instability. The capillary pressure gradients do, indeed, pump fluid from thinner toward thicker regions. On the other hand, when the contact lines are fixed, the disturbances produce surface corrugations that lead to capillary pressures opposing kinematic instabilities in the same way as it does in the case of two-dimensional waves in film flow.

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